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PERMUTABILITY OF MINIMAL SUBGROUPS AND p-NILPOTENCY OF FINITE GROUPS

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ABSTRACT

In this paper it is proved that if p is a prime dividing the order of a group G with $(|G|, p-1) = 1$ and P a Sylow p-subgroup of G, then G is p-nilpotent if every subgroup of $P \cap G^{\mathcal{N}}$ of order p is permutable in $N_G(P)$ and when $p = 2$ either every cyclic subgroup of $P \cap G^{\mathcal{N}}$ of order 4 is permutable in $N_G(P)$ or P is quaternion-free. Some applications of this result are given.

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1. Introduction

All groups considered in this paper are finite. The relationship between the properties of minimal subgroups of a group G and the structure of G has been extensively studied by a number of authors (for example, see [ABP], [BG], [BU] and [GS]). A well known result is a lemma by Itô [H, p. 435], which states that a group G is p-nilpotent if every element of G of order p lies in $Z(G)$ and, when $p = 2$, every element of G of order 4 also lies in $Z(G)$, where p is a prime dividing the order of a group G.

Buckley showed in 1970 that a group G of odd order is supersolvable if all minimal subgroups of G are normal in G [BU]. Since then, there are a number of papers in the literature dealing with the generalizations of this result. In this paper, we focus on the p-nilpotence of groups which relates with Burnside's well-known theorem for p-nilpotence, that is, if p is a prime dividing the order of a group G and P is a Sylow p-subgroup of G such that P is contained in the center of its normalizer, then G is p-nilpotent.

Inspired by the above Burnside's theorem and Itô's lemma, one might wonder whether a group G is p-nilpotent if every element of G with order p lies in the center of $N_G(P)$ and every element of G with order 4 is also in the center of $N_G(P)$ when $p = 2$, where P is a Sylow p-subgroup of G. Concerning this aspect, Ballester-Bolinches and Guo have recently given an answer to this question [BG]. They proved the following result.

THEOREM A ([BG, Theorem 1 and Theorem 2]): *Let p be a prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If every element of $P \cap G'$ with order p lies in the center of $N_G(P)$ and when $p = 2$ either every element of $P \cap G'$ with order 4 lies in the center of $N_G(P)$ or P is quaternion-free and $N_G(P)$ is 2-nilpotent, then G is p-nilpotent, where G' is the commutator subgroup of G.*

Now recall that a subgroup H of a group G is permutable (or quasinormal) in G if $HK = KH$ for any subgroup K of G. It is clear that permutability is a weak form of normality. Let us denote by $G^{\mathcal{N}}$ the nilpotent residual of a group G. Since $G^{\mathcal{N}} \leq G'$, we may ask whether the group G is p-nilpotent or not if every subgroup of $P \cap G^{\mathcal{N}}$ with order p is permutable in $N_G(P)$ and, when $p = 2$, every cyclic subgroup of $P \cap G^{\mathcal{N}}$ with order 4 is permutable in $N_G(P)$, where P a Sylow p-subgroup of G. The best result we can obtain is the following:

MAIN THEOREM: *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and let P be a Sylow p-subgroup of G. If every subgroup of*

$P \cap G^{\mathcal{N}}$ with order p is permutable in $N_G(P)$ and, when $p = 2$, either every cyclic subgroup of $P \cap G^{\mathcal{N}}$ with order 4 is permutable in $N_G(P)$ or P is quaternion-free, then G is p-nilpotent.

Using this result, we may generalize Buckley's result [BU].

It can be easily seen that the hypothesis stating that when $p = 2$ either every cyclic subgroup of $P \cap G^{\mathcal{N}}$ with order 4 is permutable in $N_G(P)$ or P is quaternion-free in our main theorem cannot be removed. For example, if we let

$$A = \langle a, b | a^4 = 1, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$$

be a quaternion group, then A has an automorphism α of order 3. Let $G = \langle \alpha \rangle \ltimes A$. Then, it is clear that every element of G with order 2 lies in the center of G, but G itself is not 2-nilpotent.

We also observe that the assumption $(|G|, p-1) = 1$ cannot be removed in our main theorem. In fact, if we let $G = S_4$ be the symmetric group of degree 4, then it is clear that every subgroup of $P \cap G' = P \cap G^{\mathcal{N}}$ with order 3 is permutable in $N_G(P)$, where P is a Sylow 3-subgroup of G. But G itself is not 3-nilpotent. Even if we assume that G is a group of odd order and every subgroup of $P \cap G^{\mathcal{N}}$ with order p is normal in G, we still cannot obtain that G is p-nilpotent if we remove the assumption $(|G|, p-1) = 1$. In fact, let G be a non-cyclic group of order 21 and $p = 7$. Then every subgroup of G with order 7 is normal in G. But G is not 7-nilpotent.

2. Preliminary results

Recall that if $\mathcal{F}$ is a formation, then the $\mathcal{F}$-residual of a group G is the smallest normal subgroup $G^{\mathcal{F}}$ of G such that $G/G^{\mathcal{F}}$ is in $\mathcal{F}$. Therefore the nilpotent residual $G^{\mathcal{N}}$ of a group G is the smallest normal subgroup of G such that $G/G^{\mathcal{N}}$ is nilpotent.

First we prove the following elementary lemma, which is needed later.

LEMMA 2.1: *Let G be a 2-group. If every cyclic subgroup of G of order 2 or 4 is permutable in G, then the exponent of $\Omega_2(G)$ is at most 4.*

Proof: Let x and y be elements of G with $|x| \leq 4$ and $|y| \leq 4$. If $|x| = |y| = 2$, then it is clear that $xy = yx$ and $|xy| = 2$. Let $|x| = 2$ and $|y| = 4$. Then we have $\langle x \rangle \langle y \rangle = \langle y \rangle \langle x \rangle$. If $[x, y] \neq 1$, then $x^{-1}yx = y^{-1}$ and hence $(xy)^2 = 1$. It follows by the hypotheses that the subgroup $\langle x \rangle \langle y \rangle = \langle x \rangle \langle xy \rangle$ is of order 4, which implies that $[x, y] = 1$, a contradiction. Thus $xy = yx$ and $|xy| = 4$. Now suppose that

$|x| = |y| = 4$. By the above proof we know that $\langle x^2, y^2\rangle \leq Z(\langle x\rangle\langle y\rangle)$. It is clear that $\langle x\rangle\langle y\rangle/\langle x^2, y^2\rangle$ has exponent 2 and $\langle x^2, y^2\rangle$ has exponent 2. Hence $|xy| \leq 4$ and the exponent of $\Omega_2(G)$ is at most 4. ∎

Next we prove the following lemma, which, in fact, is a part of our main theorem.

LEMMA 2.2: *Let p be a prime dividing the order of a group G such that $(|G|, p-1) = 1$. Let P be a Sylow p-subgroup of G. If every subgroup of $P \cap G^{\mathcal{N}}$ of order p is permutable in G and, when $p = 2$, either every cyclic subgroup of $P \cap G^{\mathcal{N}}$ of order 4 is also permutable in G or P is quaternion-free, then G is p-nilpotent.*

Proof: Suppose that the result is false and let G be a minimal counterexample. It is clear that the hypotheses of the lemma are inherited by subgroups. Therefore, G is a minimal non-p-nilpotent group. By a result of Itô [R, Theorem 10.3.3], G is a minimal non-nilpotant group. It is well-known that G is of order $p^\alpha q^\beta$, where q is a prime, $q \neq p$, P is normal in G and any Sylow q-subgroup Q of G is cyclic. Moreover, $P = G^{\mathcal{N}}$ and P is of exponent p when p is odd and of exponent at most 4 when $p = 2$ (see [R, Theorem 9.1.9 and Exercises 9.11] for details).

Assume that either p is odd or $p = 2$ and every cyclic subgroup of order 2 or 4 of P is permutable in G. Then G is supersolvable. Therefore, if $p = 2$, we have that G is 2-nilpotent, a contradiction. Hence p is odd. This implies that Q centralises every subgroup of P because q does not divide $p-1$. Consequently, G is nilpotent, a contradiction

Now assume that $p = 2$ and P is quaternion-free. Then $P \cap Z(G) = 1$ by [D, Theorem 2.8]. This is a contradiction, because every element of order 2 in $Z(P)$ is also in $Z(G)$. The proof of the lemma is now complete. ∎

LEMMA 2.3 ([ABP, Lemma 2]): *Let $\mathcal{F}$ be a saturated formation. Assume that G is a group such that G does not belong to $\mathcal{F}$ and there exists a maximal subgroup M of G such that $M \in \mathcal{F}$ and $G = MF(G)$, where $F(G)$ is the Fitting subgroup of G. Then $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G, $G^{\mathcal{F}}$ is a p-group for some prime p, $G^{\mathcal{F}}$ has exponent p if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, $G^{\mathcal{F}}$ is either an elementary abelian group or $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group.*

3. Main theorem

We now establish our main theorem for p-nilpotent groups.

THEOREM 3.1: *Let p be a prime number dividing the order of a group G with $(|G|, p-1) = 1$ and P a Sylow p-subgroup of G. If every subgroup of $P \cap G^{\mathcal{N}}$ with order p is permutable in $N_G(P)$ and, when $p = 2$, either every cyclic subgroup of $P \cap G^{\mathcal{N}}$ with order 4 is permutable in $N_G(P)$ or P is quaternion-free, then G is p-nilpotent.*

Proof: Assume that the theorem is false and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then we may choose a minimal normal subgroup N of G such that N is contained in $O_{p'}(G)$. Now consider the quotient group G/N. Note that G/N satisfies the hypotheses of our theorem. The minimality of G implies that G/N is p-nilpotent and hence G is p-nilpotent, a contradiction.

(2) For every subgroup M of G satisfying $P \leq M < G$, M must be p-nilpotent. In particular, $N_G(P)$ is p-nilpotent.

Let M be a subgroup of G with $P \leq M < G$. Since $N_M(P) \leq N_G(P)$ and $P \cap M^{\mathcal{N}} \leq P \cap G^{\mathcal{N}}$, we know that M satisfies the hypotheses of our theorem. Then, by the choice of G, M is p-nilpotent. If $N_G(P) = G$, then, by Lemma 2.2, G is p-nilpotent. Hence $N_G(P) < G$ and therefore the claim (2) holds.

(3) G is solvable. Furthermore, P is a maximal subgroup of G and a Hall p'-subgroup of G is an elementary abelian q-group Q for some prime q.

Since G is not p-nilpotent, by Frobenius' theorem [R, Theorem 10.3.2], there exists a subgroup H of P such that $N_G(H)$ is not p-nilpotent. By using our claim (2), we may choose a subgroup H of P such that $N_G(H)$ is not p-nilpotent but $N_G(K)$ is p-nilpotent for every subgroup K of P with $H < K \leq P$. Now we show that $N_G(H) = G$. Suppose on the contrary that $N_G(H) < G$. Then, we have $H < P^* \leq P$ for some $P^* \in \mathrm{Syl}_p(N_G(H))$. Since $P^* \cap (N_G(H))^{\mathcal{N}} \leq P \cap G^{\mathcal{N}}$, we see that every minimal subgroup of $P^* \cap (N_G(H))^{\mathcal{N}}$ is permutable in P^* and, in addition, every cyclic subgroup of order 4 of $P^* \cap (N_G(H))^{\mathcal{N}}$ is permutable in P^* when $p = 2$ and every cyclic subgroup of order 4 of $P \cap G^{\mathcal{N}}$ is permutable in $N_G(P)$. On the other hand, by the choice of H, we know that $N_G(P^*)$ is p-nilpotent and therefore $N_{N_G(H)}(P^*)$ is p-nilpotent. It follows that $N_G(H)$ satisfies the hypotheses of our theorem for its Sylow p-subgroup P^*. Now, by the minimality of G, we immediately see that $N_G(H)$ is p-nilpotent, a contradiction. Hence $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for every subgroup K of P with

$O_p(G) < K \leq P$. Now, using Frobenius' theorem [R, Theorem 10.3.2] again, we see that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable. By the odd order theorem [FT], it follows that G is solvable.

Let $T/O_p(G)$ be a chief factor of G. Then $T/O_p(G)$ is an elementary abelian q-group for some prime $q \neq p$ and there exists a Sylow q-subgroup Q of T such that $T = QO_p(G)$. It is clear that $PT = PQ$. If $PT < G$, then, by claim (2), PT is p-nilpotent and therefore $Q \leq C_G(O_p(G))$, which contradicts the fact $C_G(O_p(G) \leq O_p(G)$ [R, Theorem 9.3.1]. Hence $G = PQ$ and Q is a Hall p'-subgroup of G. The minimality of $T/O_p(G)$ implies that $P/O_p(G)$ is a maximal subgroup of $G/O_p(G)$ and therefore P is a maximal subgroup of G. Thus (3) holds.

(4) $1 \neq P \cap G^{\mathcal{N}}$ is normal in G.

Since G is solvable and G is not p-nilpotent, we know that $1 \neq G^{\mathcal{N}} < G$. It follows from (1) that $P \cap G^{\mathcal{N}} \neq 1$. By step (3) we have that $QO_p(G)$ is normal in G and $G/QO_p(G)$ is nilpotent. Therefore $G^{\mathcal{N}} \leq QO_p(G)$ and so $P \cap G^{\mathcal{N}} = O_p(G) \cap G^{\mathcal{N}}$ is normal in G. Thus (4) holds.

(5) $G = (P \cap G^{\mathcal{N}})L$, where $L = \langle a \rangle \ltimes Q$ is a non-abelian split extension of a normal Sylow q-subgroup Q by a cyclic p-subgroup $\langle a \rangle$, $a^p \in Z(L)$ and the action of a (by conjugation) on Q is irreducible.

By the definition of $G^{\mathcal{N}}$, we know that $G/P \cap G^{\mathcal{N}}$ is p-nilpotent. Let $D/P \cap G^{\mathcal{N}}$ be a normal p-complement of $G/P \cap G^{\mathcal{N}}$. By Schur–Zassenhaus' theorem we may assume that $D = (P \cap G^{\mathcal{N}})Q$.

Let $P_1/P \cap G^{\mathcal{N}}$ be a maximal subgroup of $P/P \cap G^{\mathcal{N}}$. Then $P \leq N_G(P_1)$. The maximality of P implies that $N_G(P_1) = P$ or G. If $N_G(P_1) = P$, then $N_H(P_1) = P_1$, where $H = P_1D = P_1Q$. It is clear that $P_1 \cap H^{\mathcal{N}} \leq P \cap G^{\mathcal{N}}$ and therefore H satisfies the hypotheses of our theorem. By the minimality of G, we have that H is p-nilpotent. It follows that $(P \cap G^{\mathcal{N}})Q = (P \cap G^{\mathcal{N}}) \times Q$ and therefore Q is a normal subgroup of G, a contradiction. Hence P_1 is normal in G. It follows that $O_p(G) = P_1$ and $P/P \cap G^{\mathcal{N}}$ is a cyclic group. On the other hand, by the Frattini argument we have that

$$G = (P \cap G^{\mathcal{N}})N_G(Q).$$

Thus, noticing that P is not normal in G, we may assume that $G = (P \cap G^{\mathcal{N}})L$, where $L = \langle a \rangle \ltimes Q$ is a non-abelian split extension of a normal Sylow q-subgroup Q by a cyclic p-subgroup $\langle a \rangle$. Since $[P : O_p(G)] = p$ and $O_p(G) \cap N_G(Q) \lhd N_G(Q)$, we see that $a^p \in Z(L)$. Also, since P is a maximal subgroup of G, we know that $(P \cap G^{\mathcal{N}})Q/P \cap G^{\mathcal{N}}$ is a minimal normal subgroup of $G/P \cap G^{\mathcal{N}}$ and therefore the action of a (by conjugation) on Q is irreducible. The claim (5) is proved.

(6) If $\Omega_1(P\cap G^{\mathcal{N}})\cap\langle a\rangle=1$, then $[\Omega_1(P\cap G^{\mathcal{N}}),Q]=1$.

In fact, let $G_1=\Omega_1(P\cap G^{\mathcal{N}})L$. Then it is clear that $\Omega_1(P\cap G^{\mathcal{N}})$ is an elementary abelian group by hypothesis. Since, for any $x\in\Omega_1(P\cap G^{\mathcal{N}})$, $\langle x\rangle\langle a\rangle=\langle a\rangle\langle x\rangle$ by hypothesis, we have $x^a\in\Omega_1(P\cap G^{\mathcal{N}})\cap(\langle x\rangle\langle a\rangle)=\langle x\rangle$. This means that a induces a power automorphism of p-power order in the elementary abelian p-group $\Omega_1(P\cap G^{\mathcal{N}})$. Hence $[\Omega_1(P\cap G^{\mathcal{N}}),a]=1$. If there exist an element $1\neq x\in\Omega_1(P\cap G^{\mathcal{N}})$ and an element $1\neq g\in Q$ such that $x^g=x_1\neq x$, then we have $x^{a^{-1}ga}=x_1$ and therefore $x^{a^{-1}gag^{-1}}=x$. It follows that $C_{G_1}(x)\geq\langle\Omega_1(P\cap G^{\mathcal{N}}),\langle a\rangle,a^{-1}gag^{-1}\rangle$. Since the action of a on Q is irreducible, we have that $Q\Omega_1(P\cap G^{\mathcal{N}})/\Omega_1(P\cap G^{\mathcal{N}})$ is a minimal normal subgroup of $G_1/\Omega_1(P\cap G^{\mathcal{N}})$ and therefore $\Omega_1(P\cap G^{\mathcal{N}})\langle a\rangle$ is a maximal subgroup of G_1. It follows that $C_{G_1}(x)=\Omega_1(P\cap G^{\mathcal{N}})\langle a\rangle$ or G_1. But $1\neq a^{-1}gag^{-1}\in Q$. Hence we have $C_{G_1}(x)=G_1$, in contradiction to $x^g\neq x$. Hence $[\Omega_1(P\cap G^{\mathcal{N}}),Q]=1$ and the claim (6) is true.

(7) The final contradiction.

For the sake of convenience, we consider the following two cases:

Case 1: $p>2$ or $p=2$ and P is quaternion-free. For this case, we let $G_1=\Omega_1(P\cap G^{\mathcal{N}})L$. If $\Omega_1(P\cap G^{\mathcal{N}})\cap\langle a\rangle=1$, then, by (6), $[\Omega_1(P\cap G^{\mathcal{N}}),Q]=1$.

Now assume that $\Omega_1(P\cap G^{\mathcal{N}})\cap\langle a\rangle=\langle a^{p^\alpha}\rangle$. Then $\langle a^{p^\alpha}\rangle$ is a cyclic group with order p and $\langle a^{p^\alpha}\rangle\leq Z(G_1)$ since $a^p\leq Z(L)$. Consider the quotient group $G_1/\langle a^{p^\alpha}\rangle$. It is clear that $(\Omega_1(P\cap G^{\mathcal{N}})/\langle a^{p^\alpha}\rangle)\cap\langle a\rangle/\langle a^{p^\alpha}\rangle=1$ and every subgroup of $\Omega_1(P\cap G^{\mathcal{N}})/\langle a^{p^\alpha}\rangle$ of order p is permutable in $\Omega_1(P\cap G^{\mathcal{N}})\langle a\rangle/\langle a^{p^\alpha}\rangle$. Noting that $\Omega_1(P\cap G^{\mathcal{N}})\langle a\rangle/\langle a^{p^\alpha}\rangle$ is a maximal subgroup of $G_1/\langle a^{p^\alpha}\rangle$ and using the arguments as in (6), we see that $[\Omega_1(P\cap G^{\mathcal{N}})/\langle a^{p^\alpha}\rangle,Q\langle a^{p^\alpha}\rangle/\langle a^{p^\alpha}\rangle]=1$ and therefore Q stabilizes the chain of subgroups

$$1\leq\langle a^{p^\alpha}\rangle\leq\Omega_1(P\cap G^{\mathcal{N}}).$$

It follows from [G, Theorem 5.3.2] that $[\Omega_1(P\cap G^{\mathcal{N}}),Q]=1$.

By [G, Theorem 5.3.10] if $p>2$ and [D, Lemma 2.15] if $p=2$ we conclude that $[P\cap G^{\mathcal{N}},Q]=1$ and therefore Q is a normal subgroup of G, a contradiction.

Case 2: $p=2$ and every cyclic subgroup of $(P\cap G^{\mathcal{N}})$ with order 2 or 4 is permutable in $N_G(P)$. Let $G_2=\Omega_2(P\cap G^{\mathcal{N}})L$. If $\Omega_1(P\cap G^{\mathcal{N}})\cap\langle a\rangle=1$, then, by (6), we have $[\Omega_1(P\cap G^{\mathcal{N}}),Q]=1$. Now we assume that $\Omega_1(P\cap G^{\mathcal{N}})\cap\langle a\rangle=A=\langle c\rangle$ is a cyclic group of order 2. It is clear that $A\leq Z(\Omega_1(P\cap G^{\mathcal{N}}))$. Let $x\in\Omega_2(P\cap G^{\mathcal{N}})$ with order 4. By the hypotheses, we have $A\langle x\rangle=\langle x\rangle A$. If $[c,x]\neq1$, then $c^{-1}xc=x^{-1}$ and so $(xc)^2=1$. It follows that the subgroup $A\langle x\rangle=\langle xc\rangle A$ is of order 4, which implies that $[c,x]=1$, a contradiction. Hence, by Lemma 2.1, A is in the center of $\Omega_2(P \cap G^{\mathcal{N}})$ and therefore $A \leq Z(G_2)$. By the same argument as in Case 1 we have that $[\Omega_1(P \cap G^{\mathcal{N}}), Q] = 1$.

Next, we consider the quotient group $G_2/\Omega_1(P \cap G^{\mathcal{N}})$. It is clear that every minimal subgroup of $\Omega_2(P \cap G^{\mathcal{N}})/\Omega_1(P \cap G^{\mathcal{N}})$ is permutable in $\Omega_2(P \cap G^{\mathcal{N}})\langle a\rangle/\Omega_1(P \cap G^{\mathcal{N}})$. Noticing that $\Omega_2(P \cap G^{\mathcal{N}})\langle a\rangle/\Omega_1(P \cap G^{\mathcal{N}})$ is a maximal subgroup of $G_2/\Omega_1(P \cap G^{\mathcal{N}})$ and using the same arguments as in Case 1 we may have that $[\Omega_2(P \cap G^{\mathcal{N}})/\Omega_1(P \cap G^{\mathcal{N}}), Q\Omega_1(P \cap G^{\mathcal{N}})/\Omega_1(P \cap G^{\mathcal{N}})] = 1$ and therefore Q stabilizes the chain of subgroups

$$1 \leq \Omega_1(P \cap G^{\mathcal{N}}) \leq \Omega_2(P \cap G^{\mathcal{N}}).$$

By [G, Theorem 5.3.2], $[\Omega_2(P \cap G^{\mathcal{N}}), Q] = 1$. It follows from [H, Satz 4.5.12] that $[P \cap G^{\mathcal{N}}, Q] = 1$ and therefore Q is normal in G, a contradiction.

The proof of the theorem is complete. ∎

COROLLARY 3.2: *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. If every subgroup of $P \cap G^{\mathcal{N}}$ with order p is permutable in $N_G(P)$ and, when $p = 2$, either every cyclic subgroup of $P \cap G^{\mathcal{N}}$ with order 4 is permutable in $N_G(P)$ or P is quaternion-free, then G is p-nilpotent.*

4. Applications

As we have mentioned above, Buckley showed that a group G of odd order is supersolvable if each minimal subgroup of G is normal in G. Now we generalize this result. We replace not only the normal assumption of minimal subgroups by the permutable assumption of minimal subgroups, but also all minimal subgroups of G by some minimal subgroups of G. In fact, our result is more general.

THEOREM 4.1: *Let $\mathcal{F}$ be a saturated formation containing the class $\mathcal{U}$ of supersolvable groups. Let N be a normal subgroup of a group G such that G/N is in $\mathcal{F}$. If for every prime p dividing the order of N and for every Sylow p-subgroup P of N, every subgroup of prime order of $P \cap G^{\mathcal{N}}$ is permutable in $N_G(P)$ and, when $p = 2$, either every cyclic subgroup of order 4 of $P \cap G^{\mathcal{N}}$ is permutable in $N_G(P)$ or P is quaternion-free, then G is in $\mathcal{F}$.*

Proof: Assume that the theorem is false and let G be a counterexample of minimal order. By our Theorem 3.1, we know that N is a Sylow tower group of supersolvable type. Thus, if p is the largest prime dividing the order of N and

P is a Sylow p-subgroup of N, then P must be normal in G. Now, we consider the quotient group G/P. Then G/P has a normal subgroup N/P such that $(G/P)/(N/P) \simeq G/N \in \mathcal{F}$. If H/P is a Sylow q-subgroup of N/P, then $q \neq p$ and we may choose a Sylow q-subgroup Q of N such that $H = PQ$. Moreover, $(G/P)^{\mathcal{N}} = G^{\mathcal{N}}P/P$. Let $\overline{x}$ be an element of order q or 4 in $(QP/P) \cap (G/P)^{\mathcal{N}}$. Then $\overline{x} = xP$ for some element $x \in Q \cap G^{\mathcal{N}}$. By hypothesis, $\langle x \rangle$ is permutable in $N_G(Q)$. It follows that $\langle \overline{x} \rangle$ is permutable in $N_{G/P}(QP/P) = N_G(Q)P/P$. Hence G/P satisfies the hypotheses of our theorem and therefore the minimality of G implies that G/P is in $\mathcal{F}$. This also implies that every subgroup of prime order of $P \cap G^{\mathcal{N}}$ is permutable in G and, when $p = 2$, either every subgroup of order 4 of $P \cap G^{\mathcal{N}}$ is permutable in G or P is quaternion-free.

Since $G^{\mathcal{F}} \leq G^{\mathcal{N}}$ and G is not in $\mathcal{F}$, we see that $1 \neq G^{\mathcal{F}}$ is contained in $P \cap G^{\mathcal{N}}$ and $G^{\mathcal{F}}$ is a p-group. Now by [B, Theorem 3.5], there exists a maximal subgroup M of G such that $G = MF'(G)$, where $F'(G) = \mathrm{Soc}(G \bmod \Phi(G))$ and $G/\mathrm{core}_G(M)$ is not in $\mathcal{F}$. Then $G = MG^{\mathcal{F}}$ and therefore $G = MF(G)$ since $G^{\mathcal{F}}$ is a p-group, where $F(G)$ is the Fitting subgroup of G. It is now clear that M satisfies the hypotheses of our theorem for its normal subgroup $M \cap P$. Hence, by the minimality of G, we have that M belongs to $\mathcal{F}$.

Now, by Lemma 2.3, $G^{\mathcal{F}}$ has exponent p when $p \neq 2$ and exponent at most 4 when $p = 2$. If $G^{\mathcal{F}}$ is an elementary abelian group, then $G^{\mathcal{F}}$ is a minimal normal subgroup of G. It is clear that $G^{\mathcal{F}}$ is not contained in the Frattini subgroup of G. Thus there is a maximal subgroup L of G such that $G = LG^{\mathcal{F}}$ and $L \cap G^{\mathcal{F}} = 1$. For any minimal subgroup A of $G^{\mathcal{F}}$, by our hypotheses, A is permutable in $G = N_G(P)$ and therefore LA is a subgroup of G. It follows that $G^{\mathcal{F}} = A$ is a cyclic group of order p.

We now suppose that $G^{\mathcal{F}}$ is not an elementary abelian group. Then $(G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$ is an elementary abelian group by Lemma 2.3. For any minimal subgroup $\overline{A}$ of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$, there exists a subgroup A of $G^{\mathcal{F}}$ such that $\overline{A} = A(G^{\mathcal{F}})'/(G^{\mathcal{F}})'$. If A is of prime order, then, noticing that

$$(G^{\mathcal{F}}/(G^{\mathcal{F}})') \cap (\Phi(G)/(G^{\mathcal{F}})') = 1$$

and by using the above arguments, we can prove that $\overline{A}$ is normal in $G/(G^{\mathcal{F}})'$. The minimality of $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ implies that $\overline{A} = G^{\mathcal{F}}/(G^{\mathcal{F}})'$, and therefore $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group of prime order. Hence the remaining case is when $p = 2$ and every element generating $G^{\mathcal{F}}$ is of order 4. It follows immediately that $\Omega_1(G^{\mathcal{F}}) = (G^{\mathcal{F}})' = Z(G^{\mathcal{F}}) = \Phi(G^{\mathcal{F}})$. For any minimal subgroup A of $\Omega_1(G^{\mathcal{F}})$ and any Sylow q-subgroup Q of G ($q \neq 2$), by our hypotheses, $QA = AQ$ is a subgroup of $\Omega_1(G^{\mathcal{F}})Q$. It follows that A is a normal subgroup of QA and therefore $A \leq Z(AQ)$. Hence every $2'$-element of G acts (by conjugation) trivially on $\Omega_1(G^{\mathcal{F}})$. If P is quaternion-free, by [D, Lemma 2.15] every $2'$-element of G acts trivially on $G^{\mathcal{F}}$. Since $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a chief factor of G, we see that $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group of prime order. If every cyclic subgroup of order 4 of $P \cap G^{\mathcal{N}}$ is permutable in $N_G(P) = G$, then, using the above arguments, we can also prove that $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is a cyclic group of prime order. We have now shown that in any case, $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is always a cyclic group of prime order. Noticing that $G^{\mathcal{F}}/(G^{\mathcal{F}})'$ is G-isomorphic to $\mathrm{Soc}(G/\mathrm{core}_G(M))$, it follows that $G/\mathrm{core}_G(M)$ is supersolvable, a contradiction. Thus, our proof is completed. ∎

Remark 4.2: It is noted that Theorem 4.1 is not true if the saturated formation $\mathcal{F}$ does not contain $\mathcal{U}$. For example, if $\mathcal{F}$ is the saturated formation of all nilpotent groups, then the symmetric group of degree three is a counterexample.

Remark 4.3: It is also noted that Theorem 4.1 is generally not true for non-saturated formations. To see this remark, we let $\mathcal{F}$ be a formation composed by all groups G such that $G^{\mathcal{U}}$ is elementary abelian. Clearly, $\mathcal{F} \geq \mathcal{U}$, but $\mathcal{F}$ is not saturated. Let $G = SL(2,3)$ and $H = Z(G)$. Then G/H is isomorphic to the alternating group of degree four and thereby $G/H \in \mathcal{F}$. But G does not belong to $\mathcal{F}$. This illustrates the situation.

COROLLARY 4.4: *Let G be a group. If for every prime p dividing the order of G and for every Sylow p-subgroup P of G, every subgroup of prime order of $P \cap G^{\mathcal{N}}$ is permutable in $N_G(P)$ and, when $p = 2$, either every cyclic subgroup of order 4 of $P \cap G^{\mathcal{N}}$ is permutable in $N_G(P)$ or P is quaternion-free, then G is supersolvable.*

Using arguments similar to those in the proof of [BG, Theorem 3], we may prove the following

THEOREM 4.5: *Let M be a nilpotent maximal subgroup of a group G and let P be a Sylow 2-subgroup of M. If every subgroup of $P \cap G^{\mathcal{N}}$ with order 2 is permutable in P and either P is quaternion-free or every cyclic subgroup of $P \cap G^{\mathcal{N}}$ with order 4 is permutable in P, then G is solvable.*

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